

Homologically arc-homogeneous ENRs

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We prove that an arc-homogeneous Euclidean neighborhood retract is a homology manifold.

[57P05](#); [57T05](#)

1 Introduction

The so-called Modified Bing–Borsuk Conjecture, which grew out of a question in [1], asserts that a homogeneous Euclidean neighborhood retract is a homology manifold. At this mini-workshop on exotic homology manifolds, Frank Quinn asked whether a space that satisfies a similar property, which he calls *homological arc-homogeneity*, is a homology manifold. The purpose of this note is to show that the answer to this question is yes.

2 Statement and proof of the main result

Theorem 2.1 *Suppose that X is an n –dimensional homologically arc-homogeneous ENR. Then X is a homology n –manifold.*

Definitions A *homology n –manifold* is a space X having the property that, for each $x \in X$,

$$H_k(X, X - x; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n. \end{cases}$$

A *Euclidean neighborhood retract* (ENR) is a space homeomorphic to a closed subset of Euclidean space that is a retract of some neighborhood of itself. A space X is *homologically arc-homogeneous* provided that for every path $\alpha: [0, 1] \rightarrow X$, the inclusion induced map

$$H_*(X \times 0, X \times 0 - (\alpha(0), 0)) \rightarrow H_*(X \times I, X \times I - \Gamma(\alpha))$$

is an isomorphism, where $\Gamma(\alpha)$ denotes the graph of α . The *local homology sheaf* \mathcal{H}_k in dimension k on a space X is the sheaf with stalks $H_k(X, X - x)$, $x \in X$.

By a result of Bredon [2, Theorem 15.2], if an n -dimensional space X is cohomologically locally connected (over \mathbb{Z}), has finitely generated local homology groups $H_k(X, X - x)$ for each k , and if each \mathcal{H}_k is locally constant, then X is a homology manifold. We shall show that an n -dimensional, homologically arc-connected ENR satisfies the hypotheses of Bredon's theorem.

Assume from now on that X represents an n -dimensional, homologically arc-homogeneous ENR. Unless otherwise specified, all homology groups are assumed to have integer coefficients. The following lemma is a straightforward application of the definition and the Mayer-Vietoris theorem.

Lemma 2.2 *Given a path $\alpha: [0, 1] \rightarrow X$ and $t \in [0, 1]$, the inclusion induced map*

$$H_*(X \times t, X \times t - (\alpha(t), t)) \rightarrow H_*(X \times I, X \times I - \Gamma(\alpha))$$

is an isomorphism.

Given points $x, y \in X$, an arc $\alpha: I \rightarrow X$ from x to y , and an integer $k \geq 0$, let $\alpha_*: H_k(X, X - x) \rightarrow H_k(X, X - y)$ be defined by the composition

$$H_k(X, X - x) \xrightarrow{\times 0} H_*(X \times I, X \times I - \Gamma(\alpha)) \xleftarrow{\times 1} H_k(X, X - y).$$

Clearly $(\alpha^{-1})_* = \alpha_*^{-1}$ and $(\alpha\beta)_* = \beta_*\alpha_*$, whenever $\alpha\beta$ is defined.

Lemma 2.3 *Given $x \in X$ and $\eta \in H_k(X, X - x)$, there is a neighborhood U of x in X such that if α and β are paths in U from x to y , then $\alpha_*(\eta) = \beta_*(\eta) \in H_k(X, X - y)$.*

Proof We will prove the equivalent statement: for each $x \in X$ and $\eta \in H_k(X, X - x)$ there is a neighborhood U of x with $\alpha_*(\eta) = \eta$ for any loop α in U based at x .

Suppose $x \in X$ and $\eta \in H_k(X, X - x)$. Since $H_k(X, X - x)$ is the direct limit of the groups $H_k(X, X - W)$, where W ranges over the (open) neighborhoods of x in X , there is a neighborhood U of x and an $\eta_U \in H_k(X, X - U)$ that goes to η under the inclusion $H_k(X, X - U) \rightarrow H_k(X, X - x)$.

Suppose α is a loop in U based at x . Let $\eta_\alpha \in H_k(X \times I, X \times I - \Gamma(\alpha))$ correspond to η_U under the isomorphism $H_k(X, X - U) \xrightarrow{\times 0} H_k(X \times I, X \times I - \Gamma(\alpha))$ guaranteed by homological arc-homogeneity.

Let

$$\eta_{U \times I} = \eta_U \times 0 \in H_k(X \times I, X \times I - U \times I).$$

Then the image of $\eta_{U \times I}$ in $H_k(X \times I, X \times I - \Gamma(\alpha))$ is η_α , as can be seen by chasing the following diagram around the lower square.

$$\begin{array}{ccc}
 H_k(X, X - U) & \longrightarrow & H_k(X, X - x) \\
 \cong \downarrow \times 1 & & \cong \downarrow \times 1 \\
 H_k(X \times I, X \times I - U \times I) & \longrightarrow & H_k(X \times I, X \times I - \Gamma(\alpha)) \\
 \cong \uparrow \times 0 & & \cong \uparrow \times 0 \\
 H_k(X, X - U) & \longrightarrow & H_k(X, X - x)
 \end{array}$$

But from the upper square we see that η_α must also come from η after including into $X \times 1$. That is, $\alpha_*(\eta) = \eta$. \square

Corollary 2.4 Suppose the neighborhood U above is path connected and F is the cyclic subgroup of $H_k(X, X - U)$ generated by η_U . Then, for every $y \in U$, the inclusion $H_k(X, X - U) \rightarrow H_k(X, X - y)$ takes F one-to-one onto the subgroup F_y generated by $\alpha_*(\eta)$, where α is any path in U from x to y .

Lemma 2.5 Suppose $x, y \in X$ and α and β are path-homotopic paths in X from x to y . Then $\alpha_* = \beta_*: H_k(X, X - x) \rightarrow H_k(X, X - y)$.

Proof By a standard compactness argument it suffices to show that, for a given path α from x to y and element $\eta \in H_k(X, X - x)$, there is an $\epsilon > 0$ such that $\alpha_*(\eta) = \beta_*(\eta)$ for any path β from x to y ϵ -homotopic (rel $\{x, y\}$) to α .

Given a path α from x to y , $\eta \in H_k(X, X - x)$, and $t \in I$, let U_t be a path-connected neighborhood of $\alpha(t)$ associated with $(\alpha_t)_*(\eta) \in H_k(X, X - \alpha(t))$ given by Lemma 2.3, where α_t is the path $\alpha|_{[0, t]}$. There is a subdivision

$$\{0 = t_0 < t_1 < \cdots < t_m = 1\}$$

of I such that $\alpha([t_{i-1}, t_i]) \subseteq U_i$ for each $i = 1, \dots, m$, where $U_i = U_{t_i}$ for some t_i . There is an $\epsilon > 0$ so that if $H: I \times I \rightarrow X$ is an ϵ -path-homotopy from α to a path β , then $H([t_{i-1}, t_i] \times I) \subseteq U_i$.

For each $i = 1, \dots, m$, let $\alpha_i = \alpha|_{[t_{i-1}, t_i]}$ and $\beta_i = \beta|_{[t_{i-1}, t_i]}$, and for $i = 0, \dots, m$, let $\gamma_i = H|_{t_i \times I}$ and $\eta_i = (\alpha_{t_i})_*(\eta)$. By Corollary 2.4,

$$(\alpha_i)_*(\eta_{i-1}) = (\gamma_{i-1}\beta_i\gamma_i^{-1})_*(\eta_{i-1}) = \eta_i$$

where $\eta_0 = \eta$. Since γ_0 and γ_m are the constant paths, it follows easily that

$$\alpha_*(\eta) = (\alpha_m)_* \cdots (\alpha_1)_*(\eta) = (\beta_m)_* \cdots (\beta_1)_*(\eta) = \beta_*(\eta). \quad \square$$

Proof of Theorem 2.1 As indicated at the beginning of this note, we need only show that the hypotheses of [2, Theorem 15.2] are satisfied.

Since X is an ENR, it is locally contractible, and hence cohomologically locally connected over \mathbb{Z} .

Given $x \in X$, let W be a path-connected neighborhood of x such that W is contractible in X . If α and β are two paths in W from x to a point $y \in W$, then α and β are path-homotopic in X . Hence, by Lemma 2.5, $\alpha_*: H_k(X, X - x) \rightarrow H_k(X, X - y)$ is a well-defined isomorphism that is independent of α for every $k \geq 0$. Hence, $\mathcal{H}_k|_W$ is the constant sheaf, and so \mathcal{H}_k is locally constant.

Finally, we need to show that the local homology groups of X are finitely generated. This can be seen by working with a mapping cylinder neighborhood of X . Assume X is nicely embedded in \mathbb{R}^{n+m} , for some $m \geq 3$, so that X has a mapping cylinder neighborhood $N = C_\phi$ of a map $\phi: \partial N \rightarrow X$, with mapping cylinder projection $\pi: N \rightarrow X$ (see [3]). Given a subset $A \subseteq X$, let $A^* = \pi^{-1}(A)$ and $\dot{A} = \phi^{-1}(A)$.

Lemma 2.6 *If A is a closed subset of X , then $H_k(X, X - A) \cong \check{H}_c^{n+m-k}(A^*, \dot{A})$.*

Proof Suppose A is closed in X . Since $\pi: N \rightarrow X$ is a proper homotopy equivalence,

$$H_k(X, X - A) \cong H_k(N, N - A^*).$$

Since ∂N is collared in N ,

$$H_k(N, N - A^*) \cong H_k(\text{int } N, \text{int } N - A^*),$$

and by Alexander duality,

$$\begin{aligned} H_k(\text{int } N, \text{int } N - A^*) &\cong \check{H}_c^{n+m-k}(A^* - \dot{A}) \\ &\cong \check{H}_c^{n+m-k}(A^*, \dot{A}) \end{aligned}$$

(since \dot{A} is also collared in A^*). □

Since X is n -dimensional, we get the following corollary.

Corollary 2.7 *If A is a closed subset of X , then $\check{H}_c^q(A^*, \dot{A}) = 0$, if $q < m$ or $q > n + m$.*

Thus, the local homology sheaf \mathcal{H}_k of X is isomorphic to the Leray sheaf \mathcal{H}^{n+m-k} of the map $\pi: N \rightarrow X$ whose stalks are $\check{H}^{n+m-k}(x^*, \dot{x})$. For each $k \geq 0$, this sheaf is also locally constant, so there is a path-connected neighborhood U of x such that $\mathcal{H}^q|_U$ is constant for all $q \geq 0$. Given such a U , there is a path-connected neighborhood V

of x lying in U such that the inclusion of V into U is null-homotopic. Thus, for any coefficient group G , the inclusion $H^p(U, G) \rightarrow H^p(V, G)$ is zero if $p \neq 0$ and is an isomorphism for $p = 0$.

The Leray spectral sequences of $\pi|_{\pi^{-1}(U)}$ and $\pi|_{\pi^{-1}(V)}$ have E_2 terms

$$E_2^{p,q}(U) \cong H^p(U; \mathcal{H}^q), \quad E_2^{p,q}(V) \cong H^p(V; \mathcal{H}^q)$$

and converge to

$$E_\infty^{p,q}(U) \subseteq H^{p+q}(U^*, \dot{U}; \mathbb{Z}), \quad E_\infty^{p,q}(V) \subseteq H^{p+q}(V^*, \dot{V}; \mathbb{Z}),$$

respectively (see [2, Theorem 6.1]). Since the sheaf \mathcal{H}^q is constant on U and V , $H^p(U; \mathcal{H}^q)$ and $H^p(V; \mathcal{H}^q)$ represent ordinary cohomology groups with coefficients in $G_q \cong \check{H}^q(x^*, \dot{x})$.

By naturality, we have the commutative diagram

$$\begin{array}{ccc} E_2^{0,q}(U) & \longrightarrow & E_2^{2,q-1}(U) \\ \downarrow \cong & & \downarrow 0 \\ E_2^{0,q}(V) & \longrightarrow & E_2^{2,q-1}(V) \end{array}$$

which implies that the differential $d_2: E_2^{0,q}(V) \rightarrow E_2^{2,q-1}(V)$ is the zero map. Hence,

$$E_3^{0,q}(V) = \ker(E_2^{0,q}(V) \rightarrow E_2^{2,q-1}(V)) / \operatorname{im}(E_2^{-2,q+1}(V) \rightarrow E_2^{0,q}(V)) = E_2^{0,q}(V),$$

and, similarly, $E_3^{0,q}(V) = E_4^{0,q}(V) = \dots = E_\infty^{0,q}(V)$. Thus,

$$H^q(V^*, \dot{V}; \mathbb{Z}) \supseteq E_\infty^{0,q}(V) \cong E_2^{0,q}(V) \cong H^0(V; \mathcal{H}^q) \cong H^0(V; G_q) \cong G_q.$$

Applying the same argument to the inclusion $(x^*, \dot{x}) \subseteq (V^*, \dot{V})$ yields the commutative diagram

$$\begin{array}{ccc} E_2^{0,q}(V) & \longrightarrow & E_2^{2,q-1}(V) \\ \downarrow \cong & & \downarrow 0 \\ E_2^{0,q}(x) & \longrightarrow & E_2^{2,q-1}(x) \end{array}$$

which, in turn, gives

$$\begin{array}{ccc} G_q \cong H^0(V; G_q) & \xrightarrow{\cong} & H^q(V^*, \dot{V}; \mathbb{Z}) \\ \cong \downarrow & & \downarrow \\ G_q \cong H^0(x; G_q) & \xrightarrow{\cong} & H^q(x^*, \dot{x}; \mathbb{Z}) \cong G_q \end{array}$$

from which it follows that the inclusion $H^q(V^*, \dot{V}; \mathbb{Z}) \rightarrow H^q(x^*, \dot{x}; \mathbb{Z}) \cong G_q$ is an isomorphism of G_q . Since (x^*, \dot{x}) is a compact pair in the manifold pair (V^*, \dot{V}) , it has a compact manifold pair neighborhood $(W, \partial W)$. Since the inclusion $H^q(V^*, \dot{V}) \rightarrow \check{H}^q(x^*, \dot{x})$ factors through $H^q(W, \partial W)$, its image is finitely generated for each q . Hence, $H_k(X, X - x) \cong \check{H}^{n+m-k}(x^*, \dot{x})$ is finitely generated for each k . \square

The following theorem, which may be of independent interest, emerges from the proof of [Theorem 2.1](#).

Theorem 2.8 *Suppose X is an n -dimensional ENR whose local homology sheaf \mathcal{H}_k is locally constant for each $k \geq 0$. Then X is a homology n -manifold.*

References

- [1] **R H Bing, K Borsuk**, [Some remarks concerning topologically homogeneous spaces](#), Ann. of Math. (2) 81 (1965) 100–111 [MR0172255](#)
- [2] **G E Bredon**, *Sheaf theory*, McGraw-Hill Book Co., New York (1967) [MR0221500](#)
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Received: 23 March 2004